ECE317 : Feedback and Control

Lecture
Time Response

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Course roadmap

Modeling

- ✓ Laplace transform
- ✓ Transfer function
- ✓ Block Diagram Linearization
- ✓ Models for systems
  - ✓ electrical
  - ✓ mechanical
  - example system

Analysis

Stability
- Pole locations
- Routh-Hurwitz

✓ Time response
- Transient
- Steady state (error)

Frequency response
- Bode plot

Design

Design specs

Frequency domain

Bode plot

Compensation

Design examples

Matlab & PECS simulations & laboratories

Lecture Outline

Topics covered in this presentation

- Poles & zeros
- First-order systems
- Second-order systems

Chapter outline

Definitions

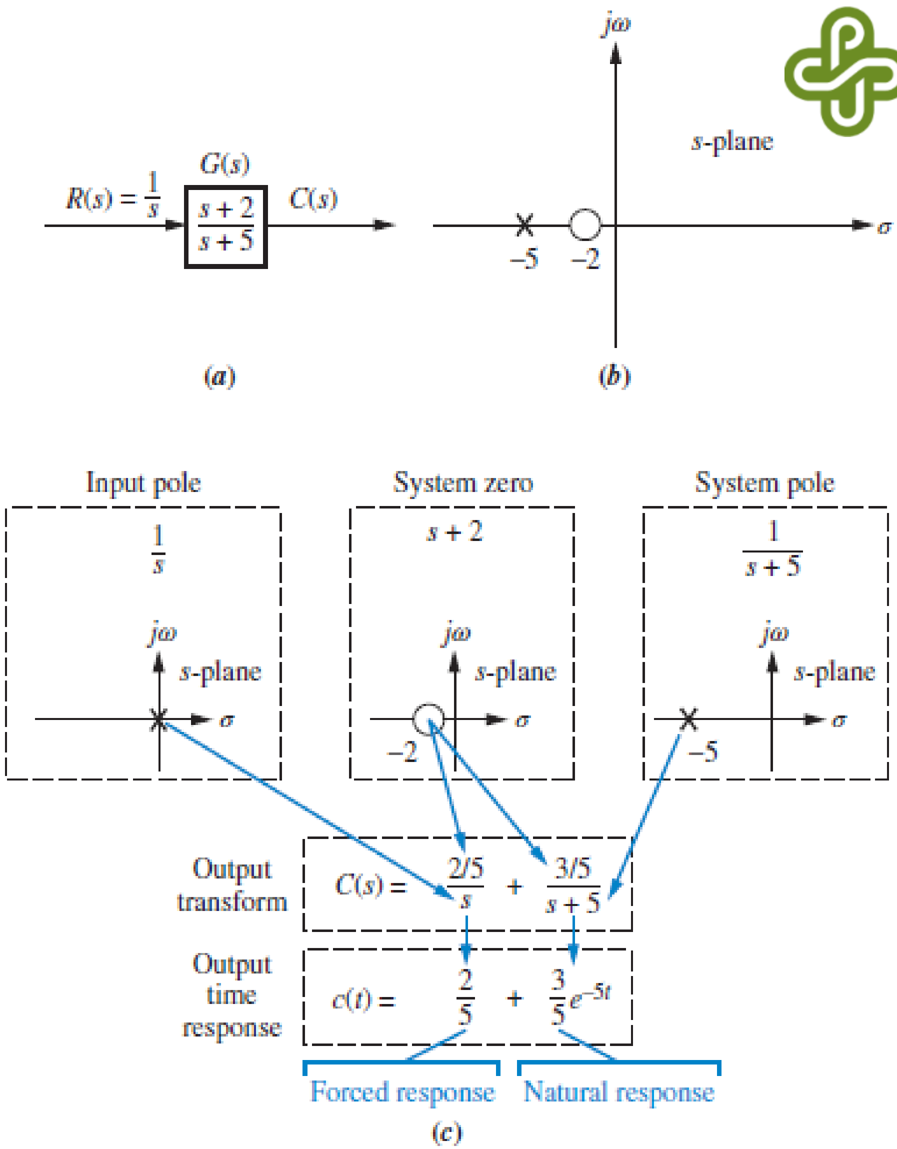


Figure: a. system showing input & output, b. pole-zero plot of the system; c. evolution of a system response

Poles of a TF

- Values of the Laplace transform variable, s, that cause the TF to become infinite
- Any roots of the denominator of the TF that are common to the roots of the numerator

Definitions

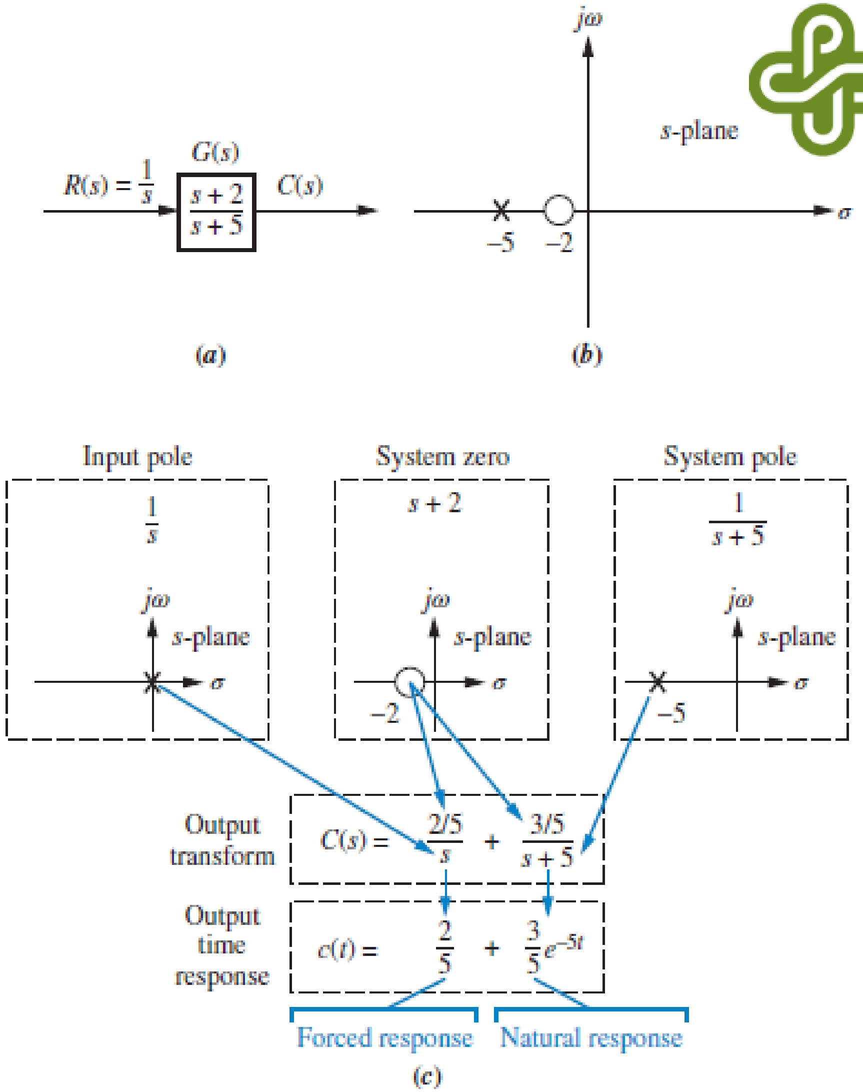


Figure: a. system showing input & output, b. pole-zero plot of the system; c. evolution of a system response

Zeros of a TF

- Values of the Laplace transform variable, s, that cause the TF to become zero
- Any roots of the numerator of the TF that are common to the roots of the denominator

System response characteristics

- *Poles of a TF:* Generate the form of the *natural response*
- *Poles of a input function:* Generate the form of the *forced response*

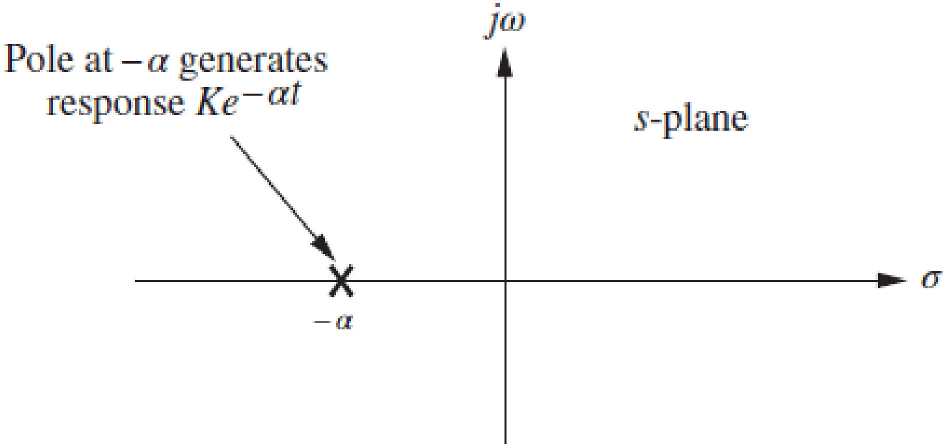


Figure: Effect of a real-axis pole upon transient response

System response characteristics

- *Pole on the real axis:* Generates an *exponential response* of the form $e^{-\alpha t}$, where $-\alpha$ is the pole location on the real axis. The farther to the left a pole is on the negative real axis, the faster the exponential transient response will decay to zero.
- *Zeros and poles:* Generate the *amplitudes* for both the forced and natural responses

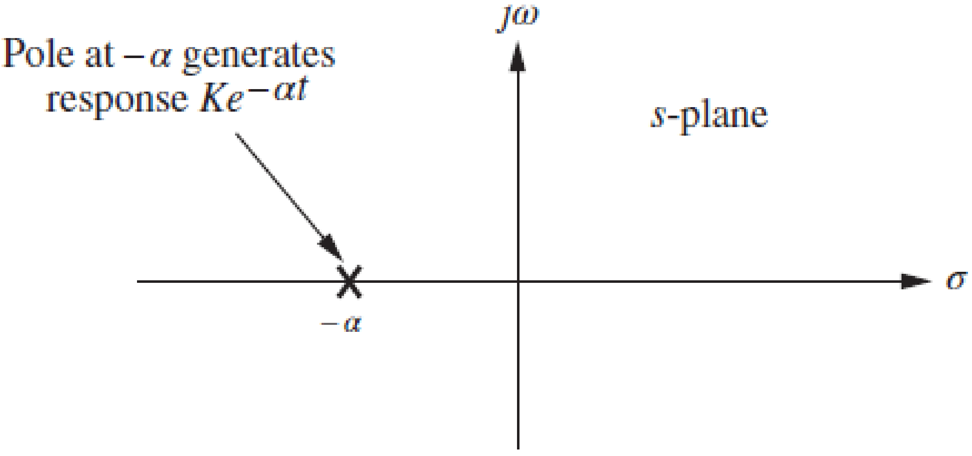


Figure: Effect of a real-axis pole upon transient response

Introduction

- 1^{st}-order system without zeros TF

$$G(s) = \frac{C(s)}{R(s)} = \frac{a}{s+a}$$

- Unit step input TF

$$R(s) = s^{-1}$$

- System response in frequency domain

$$C(s) = R(s)G(s) = \frac{a}{s(s+a)}$$

- System response in time domain

$$c(t) = c_f(t) + c_n(t) = 1 - e^{-at}$$

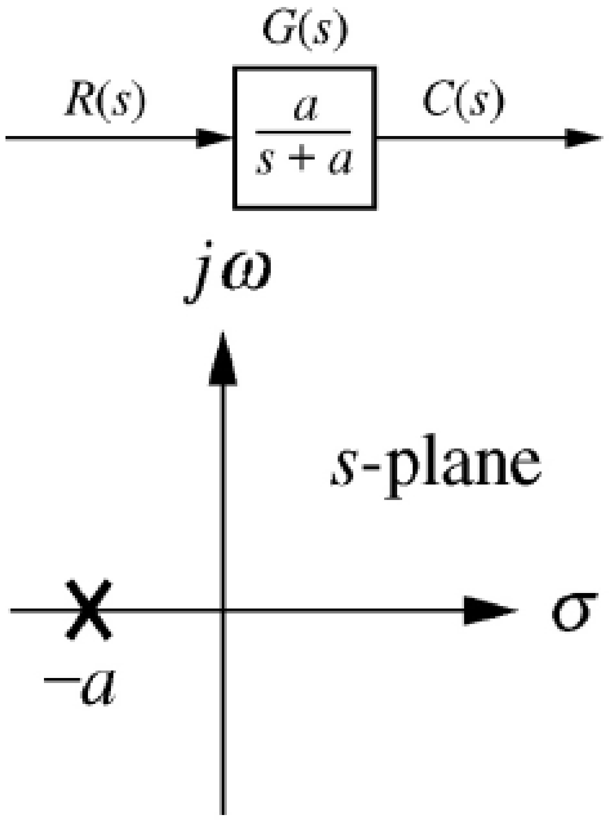


Figure: 1^{st}-order system; pole-plot

Characteristics

- *Time constant, $\frac{1}{a}$:* The time for e^{-at} to decay to 37% of its initial value. Alternatively, the time it takes for the step response to rise to 63% of its final value.

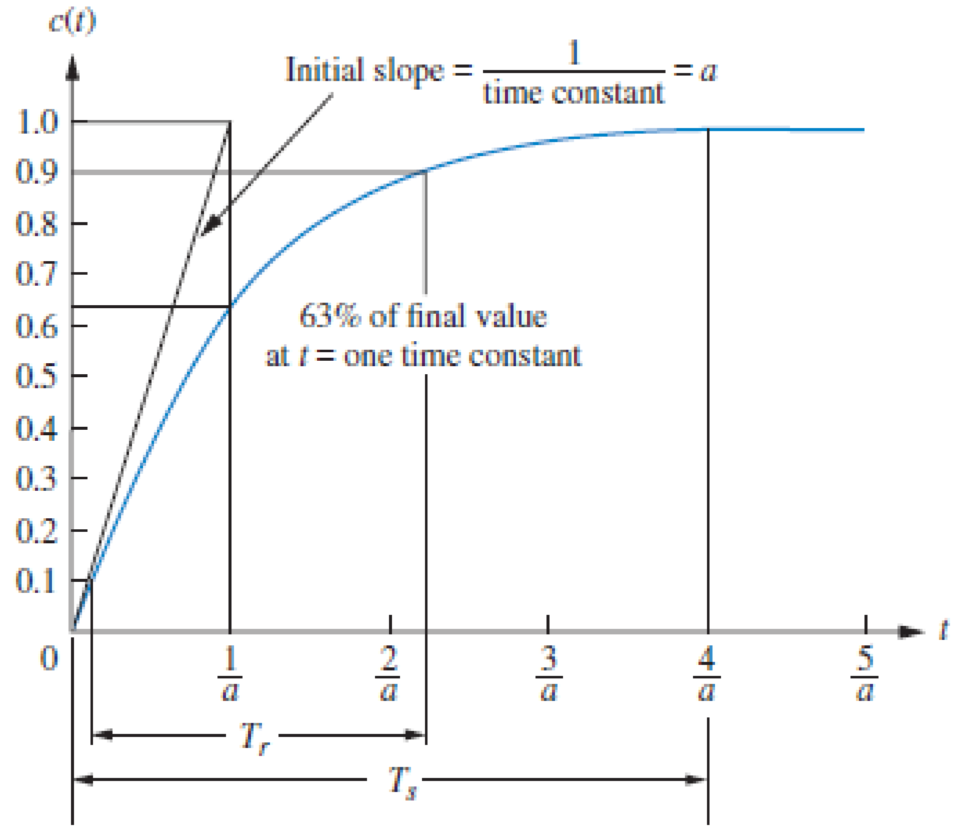


Figure: 1^{st}-order system response to a unit step

Characteristics

- *Exponential frequency, a:* The reciprocal of the time constant. The initial rate of change of the exponential at $t = 0$, since the derivative of e^{-at} is $-a$ when $t = 0$. Since the pole of the TF is at $-a$, the farther the pole is from the imaginary axis, the faster the transient response.

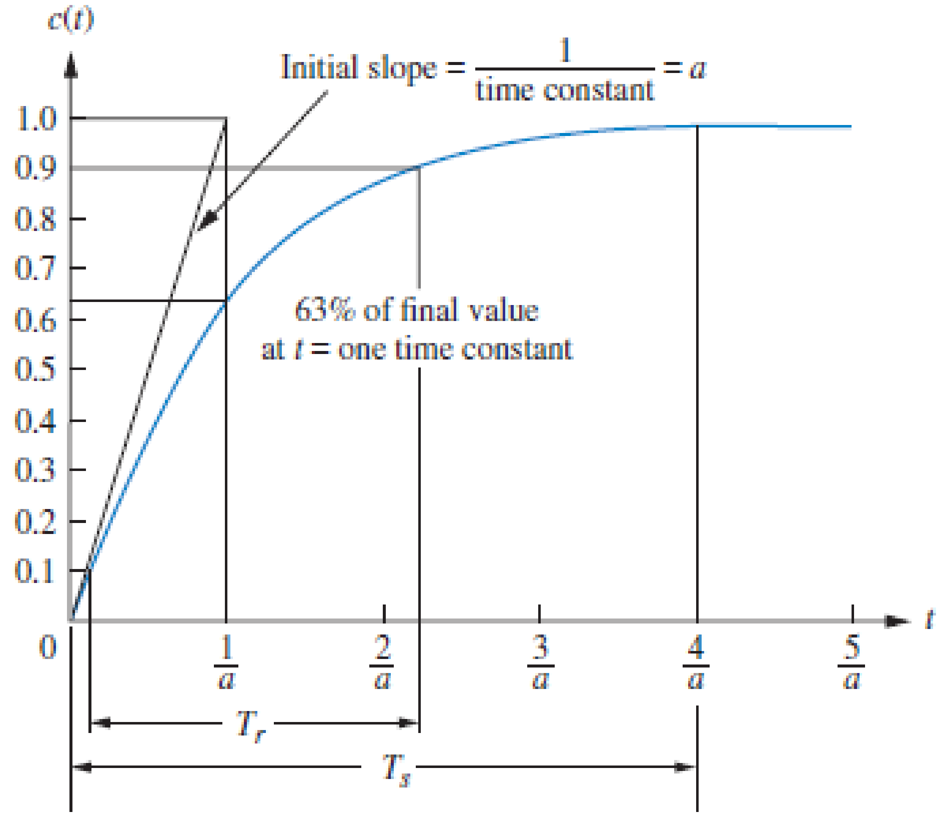


Figure: 1^{st}-order system response to a unit step

Characteristics

- *Rise time, T_r:* The time for the waveform to go from 0.1 to 0.9 of its final value. The difference in time between $c(t) = 0.9$ and $c(t) = 0.1$.

$$T_r = \frac{2.2}{a}$$

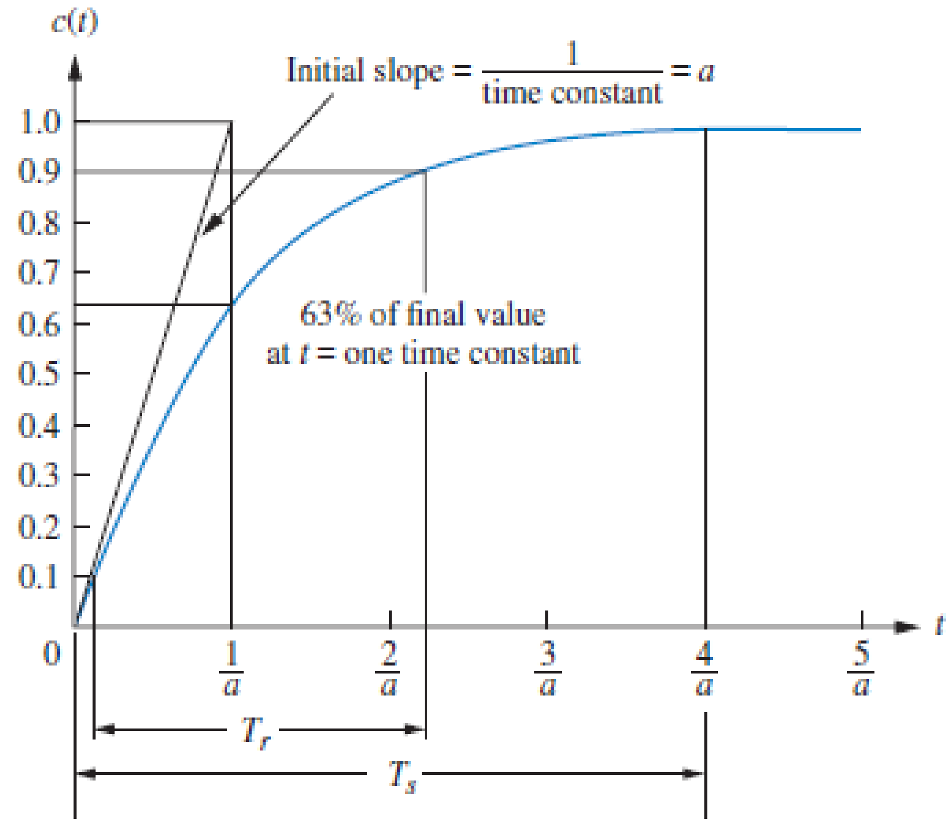


Figure: 1^{st}-order system response to a unit step

Characteristics

- *2% Settling time, T_s:* The time for the response to reach, and stay within, 2% (arbitrary) of its final value. The time when $c(t) = 0.98$.

$$T_s = \frac{4}{a}$$

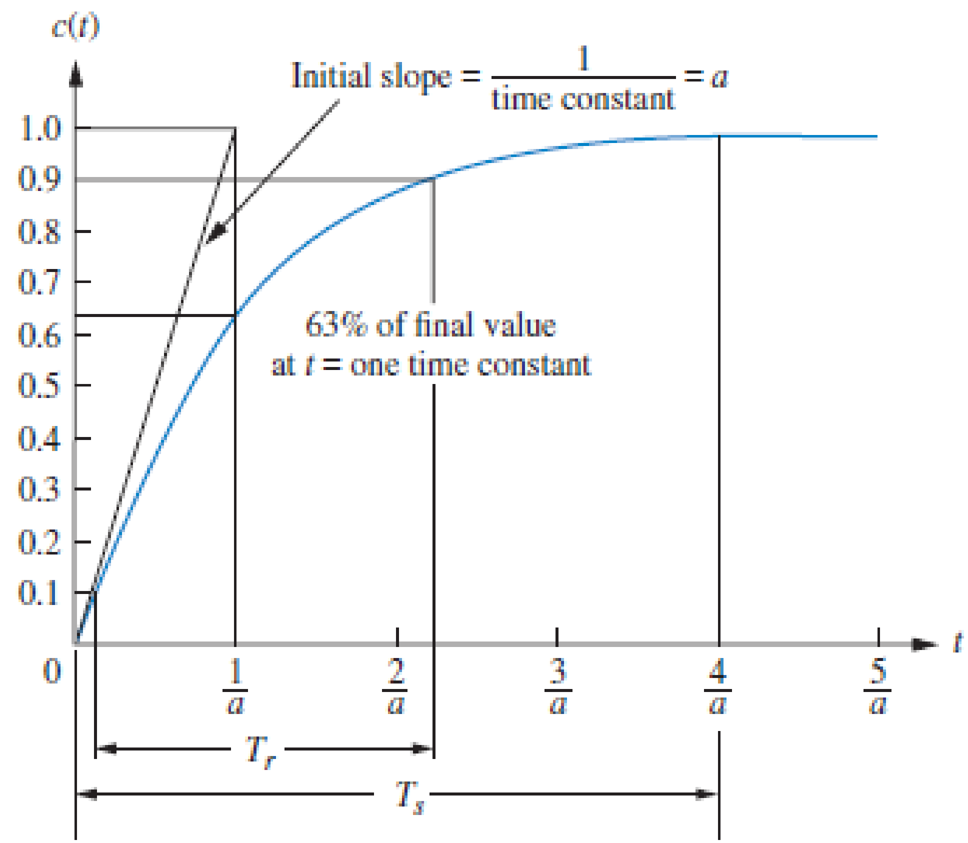


Figure: 1^{st}-order system response to a unit step

General form

- *2 finite poles:* 2 real poles or complex pole pair determined by parameters *a* and *b*
- *No zeros*

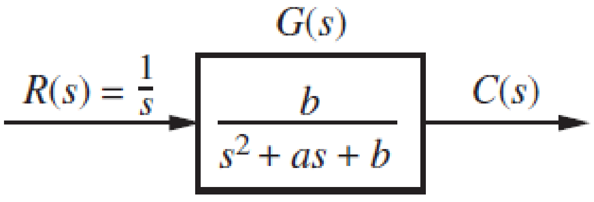


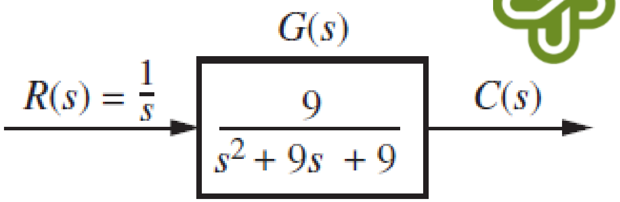
Figure: General 2^{nd}-order system

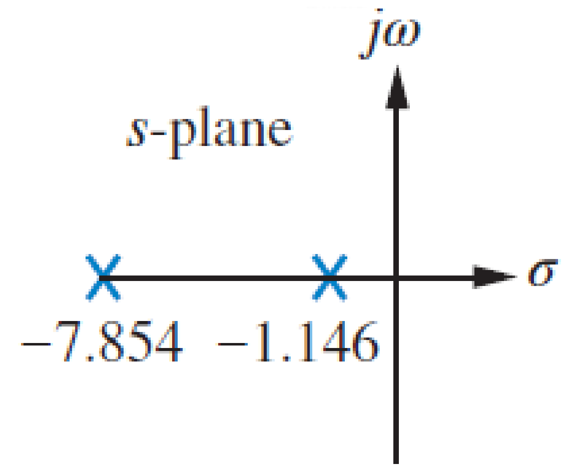
Overdamped response

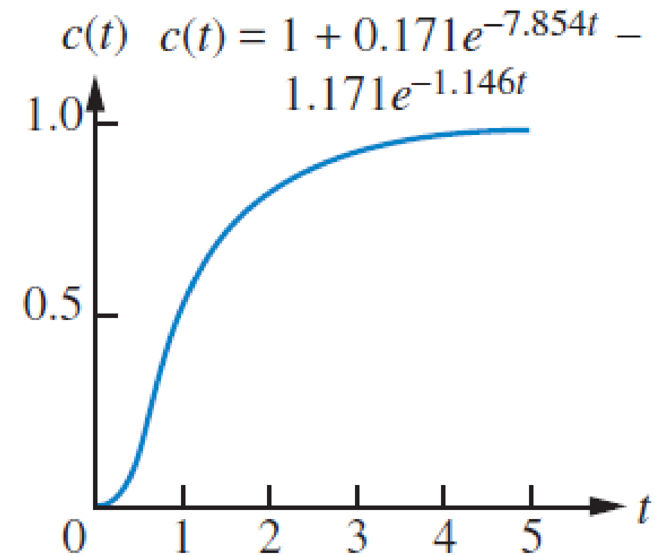
- 1 pole at origin from the unit step input
- System poles: 2 real at σ_1, σ_2
- Natural response: Summation of 2 exponentials

$$c(t) = K_1 e^{-\sigma_1 t} + K_2 e^{-\sigma_2 t}$$

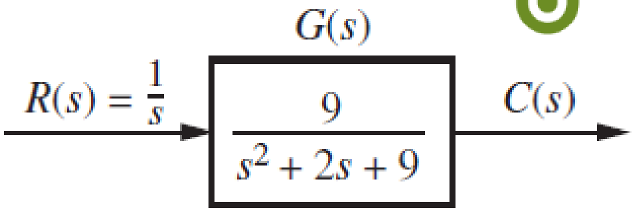
- Time constants: $-\sigma_1$, $-\sigma_2$

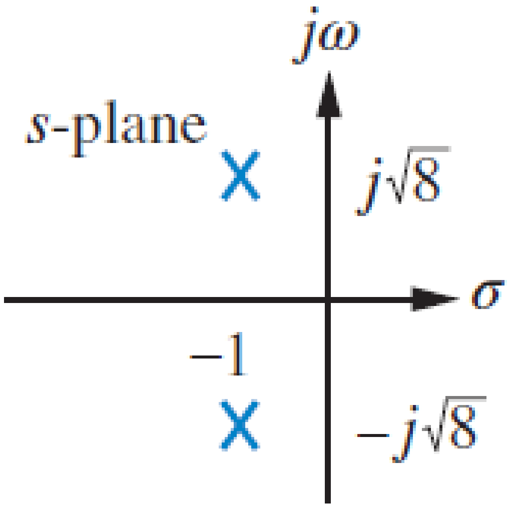






Underdamped response

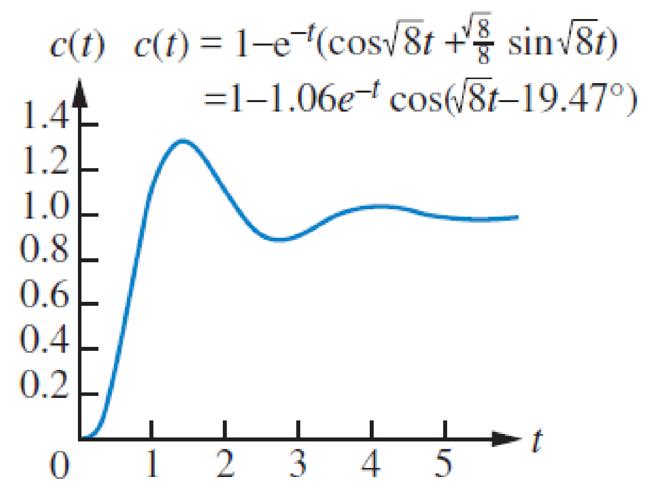




- 1 pole at origin from the unit step input
- System poles: 2 complex at $\sigma_d \pm j\omega_d$
- Natural response: Damped sinusoid with an exponential envelope

$$c(t) = K_1 e^{-\sigma_d t} cos(\omega_d t - \phi)$$

- Time constant: σ_d
- Frequency (rad/s): ω_d



Underdamped response characteristics

- *Transient response:* Exponentially decaying amplitude generated by the real part of the system pole times a sinusoidal waveform generated by the imaginary part of the system pole.
- *Damped frequency of oscillation, ω_d:* The imaginary part part of the system poles.
- *Steady state response:* Generated by the input pole located at the origin.
- *Underdamped response:* Approaches a steady state value via a transient response that is a damped oscillation.

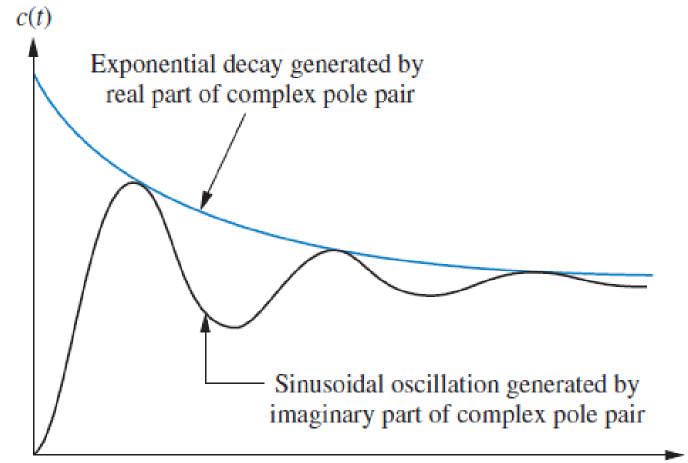
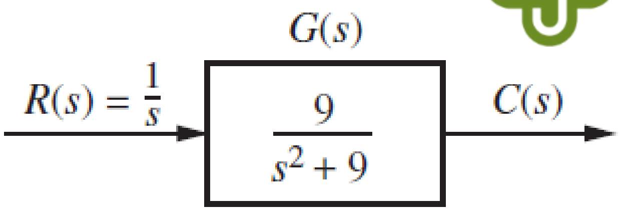
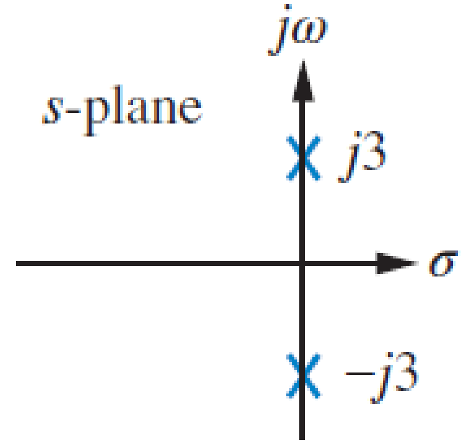
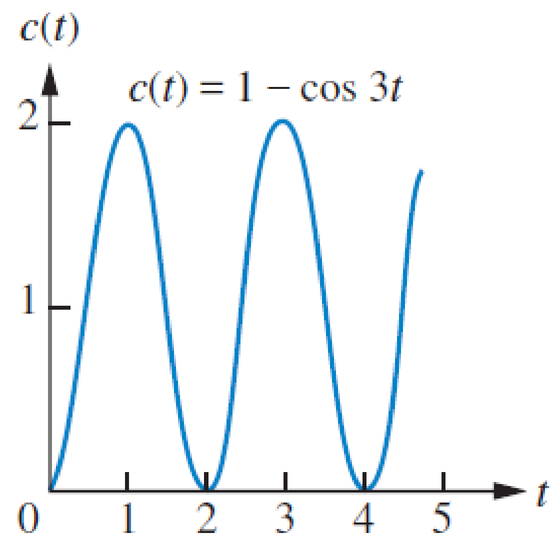


Figure: 2^{nd}-order step response components generated by complex poles

Undamped response





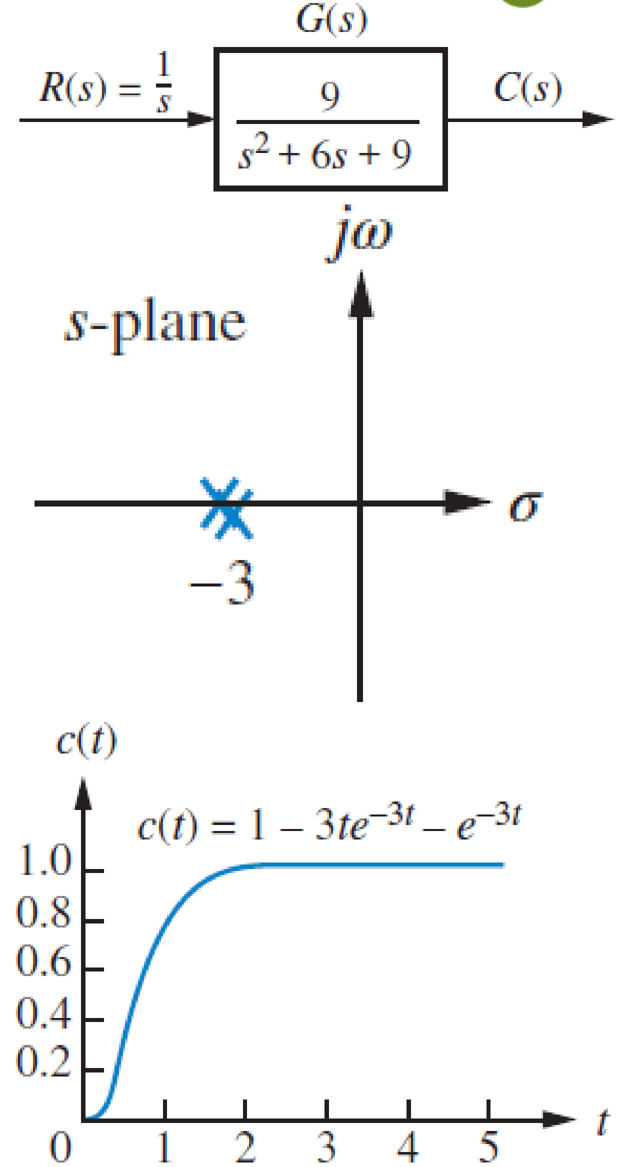


- 1 pole at origin from the unit step input
- System poles: 2 imaginary at $\pm j\omega_1$
- Natural response: Undamped sinusoid

$$c(t) = A\cos\left(\omega_1 t - \phi\right)$$

- Frequency: ω_1

Critically damped response



- 1 pole at origin from the unit step input
- System poles: 2 multiple real
- Natural response: Summation of an exponential and a product of time and an exponential

$$c(t) = K_1 e^{-\sigma_1 t} + K_2 t e^{-\sigma_1 t}$$

- Time constant: σ_1
- Note: Fastest response without overshoot

Step response damping cases

- Overdamped
- Underdamped
- Undamped
- Critically damped

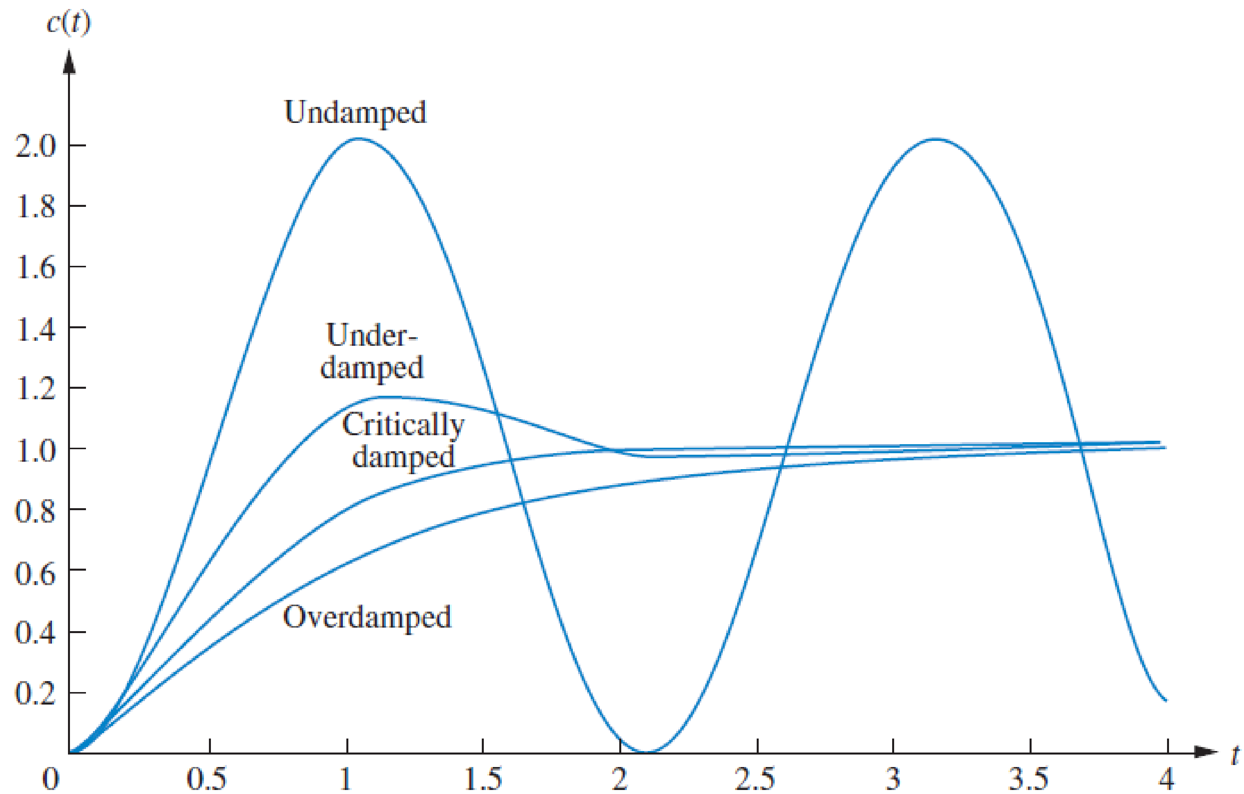


Figure: Step responses for 2^{nd}-order system damping cases

Specification

- *Natural frequency, ω_n*
  - The frequency of oscillation of the system without damping
- *Damping ratio, ζ*

$$\zeta = \frac{\text{Exponential decay frequency}}{\text{Natural frequency (rad/s)}} = \frac{1}{2\pi}\frac{\text{Natural period (s)}}{\text{Exponential time constant}}$$

- *General TF*

$$G(s) = \frac{b}{s^2 + as + b} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

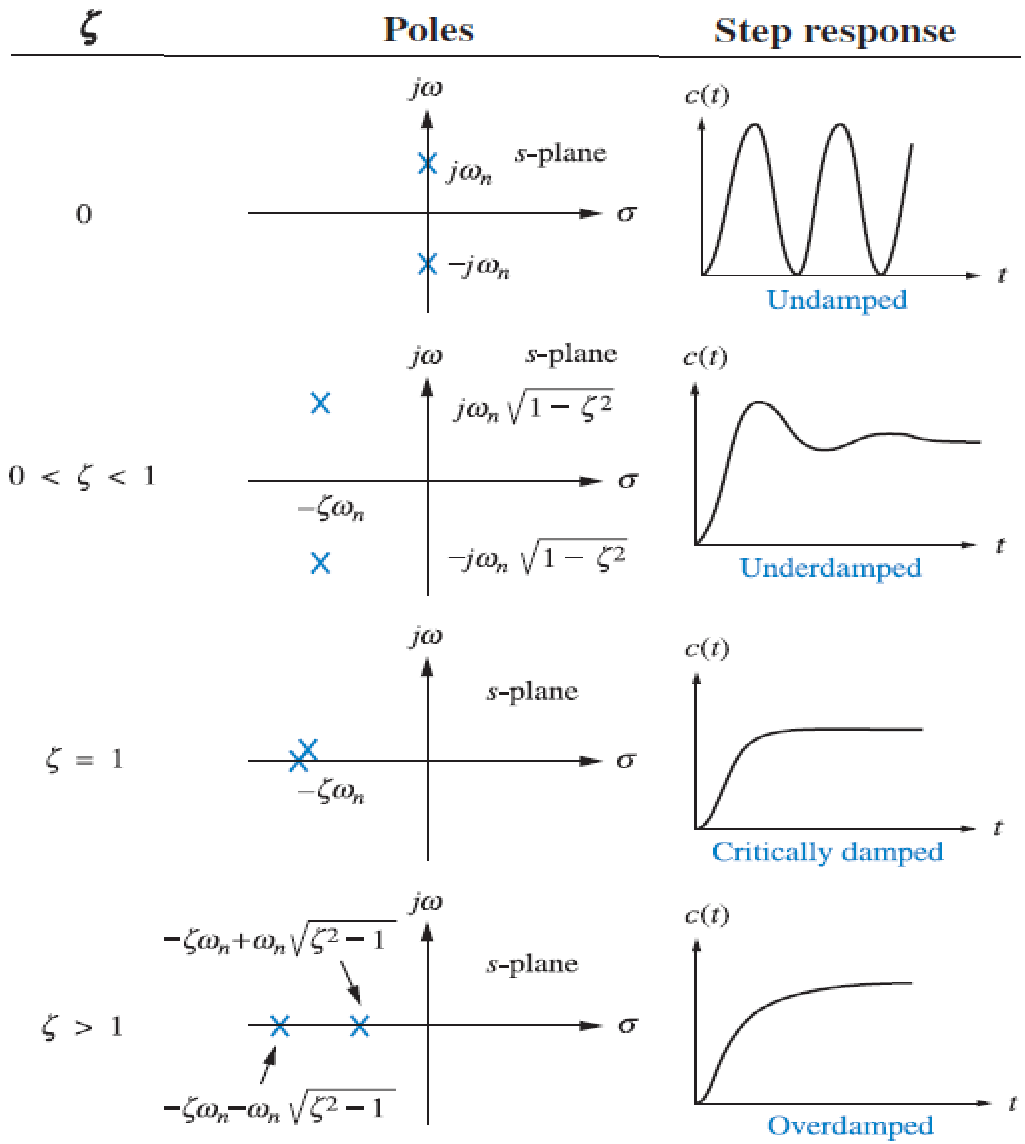
where

$$a = 2\zeta\omega_n, \quad b = \omega_n^2, \quad \zeta = \frac{a}{2\omega_n}, \quad \omega_n = \sqrt{b}$$

Response as a function of ζ

- *Poles*

$$s_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$



Step response

Output Response (Laplace domain):

$$C(s) = \frac{\omega_n^2}{s(s^2 + 2\zeta\omega_n s + \omega_n^2)}$$

...partial fraction expansion...

$$= \frac{1}{s} + \frac{(s + \zeta\omega_n) + \frac{\zeta}{\sqrt{1-\zeta^2}}\omega_n\sqrt{1-\zeta^2}}{(s + \zeta\omega_n)^2 + \omega_n^2(1-\zeta^2)}$$

Step response

Time domain via inverse Laplace transform

$$c(t) = 1 - e^{\zeta\omega_n t}\left(\cos(\omega_n\sqrt{1-\zeta^2})t + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin(\omega_n\sqrt{1-\zeta^2})t\right)$$

...trigonometry & exponential relations...

$$= 1 - \frac{1}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\cos(\omega_n\sqrt{1-\zeta^2} - \phi)$$

where

$$\phi = \tan^{-1}\left(\frac{\zeta}{\sqrt{1-\zeta^2}}\right)$$

Responses for ζ values

Response versus ζ plotted along a time axis normalized to ω_n

- Lower ζ produce a more oscillatory response
- ω_n does not affect the nature of the response other than scaling it in time

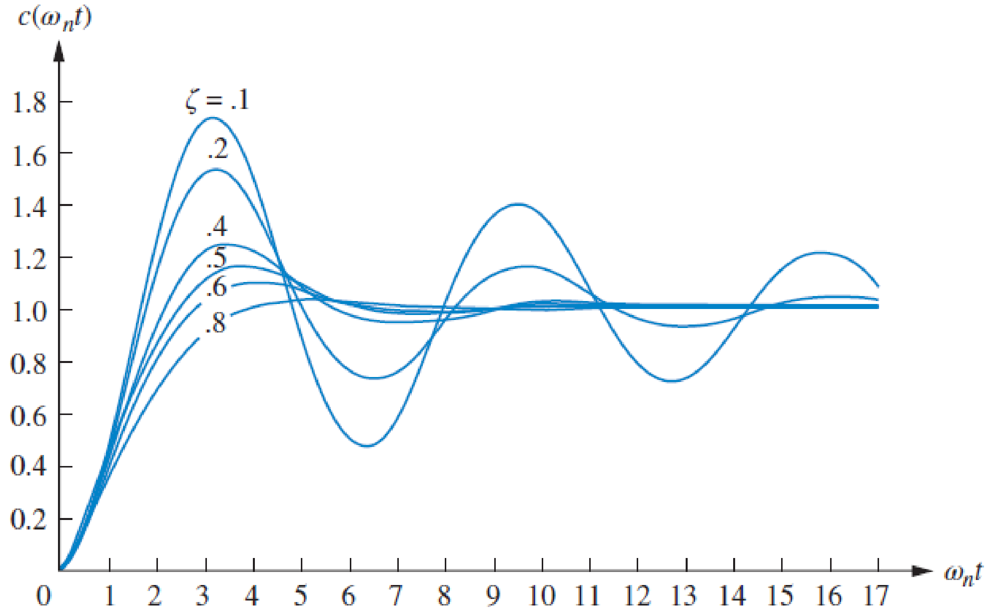


Figure: 2^{nd}-order underdamped responses for damping ratio values

Response specifications

- *Rise time, T_r:* Time required for the waveform to go from 0.1 of the final value to 0.9 of the final value
- *Peak time, T_p:* Time required to reach the first, or maximum, peak

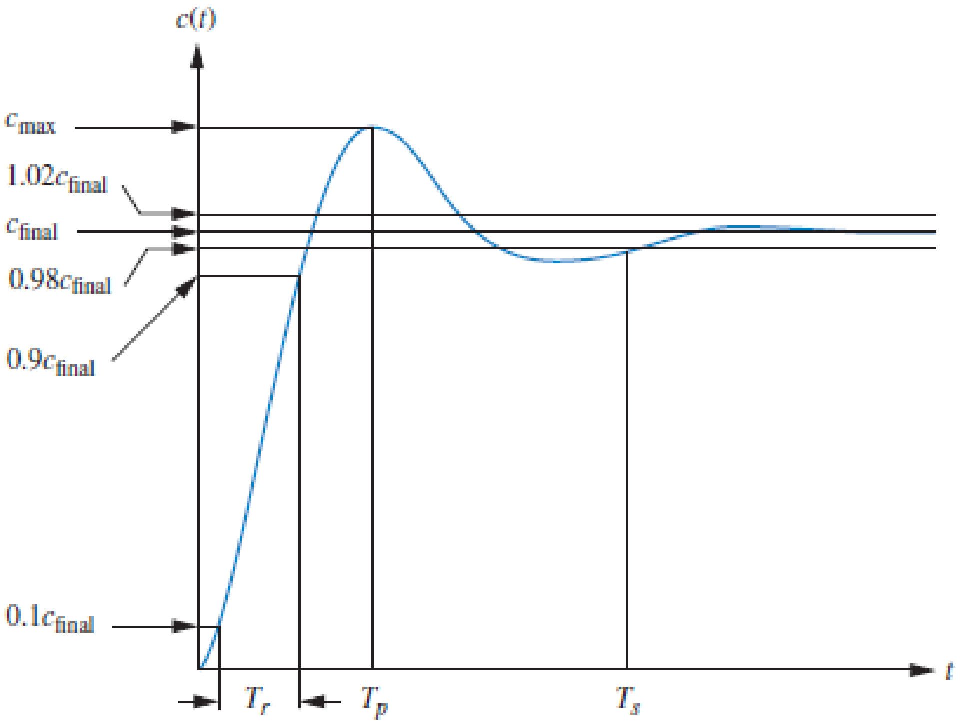


Figure: 2^{nd}-order underdamped response specifications

Response specifications

- *Overshoot, %OS:* The amount that the waveform overshoots the steady state, or final, value at the peak time, expressed as a percentage of the steady state value
- *Settling time, T_s:* Time required for the transient's damped oscillations to reach and stay within $\pm 2\%$ of the steady state value

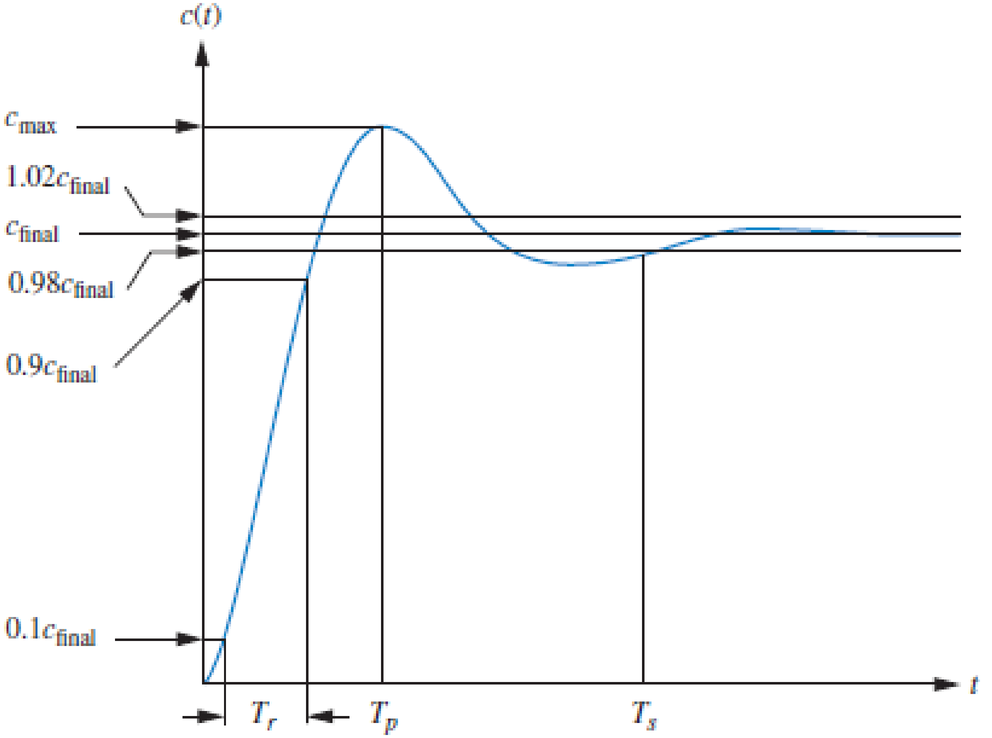


Figure: 2^{nd}-order underdamped response specifications

Evaluation of T_p

T_p is found by differentiating $c(t)$ and finding the zero crossing after $t = 0$, which is simplified by applying a derivative in the frequency domain and assuming zero initial conditions.

$$\mathcal{L}[\dot{c}(t)] = sC(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

...completing the squares in the denominator
...setting the derivative to zero

$$T_p = \frac{\pi}{\omega_n \sqrt{1 - \zeta^2}}$$

Evaluation of $\%OS$

$\%OS$ is found by evaluating

$$\%OS = \frac{c_{\text{max}} - c_{\text{final}}}{c_{\text{final}}} \times 100$$

where

$$c_{\text{max}} = c(T_p), \quad c_{\text{final}} = 1$$

...substitution

$$\%OS = e^{\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}} \times 100$$

ζ given $\%OS$

$$\zeta = \frac{-\ln(\frac{\%OS}{100})}{\sqrt{\pi^2 + \ln^2(\frac{\%OS}{100})}}$$

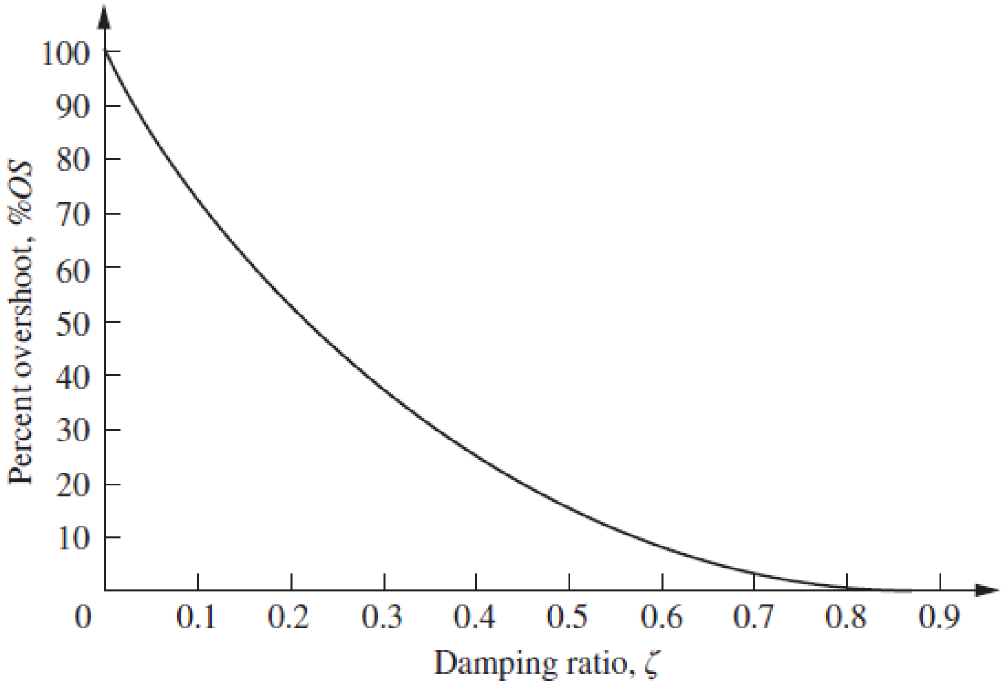


Figure: $\%OS$ vs. ζ

Evaluation of T_s

Find the time for which $c(t)$ reaches and stays within $\pm 2\%$ of the steady state value, c_{final}, i.e., the time it takes for the amplitude of the decaying sinusoid to reach 0.02

$$e^{-\zeta \omega_n t} \frac{1}{\sqrt{1-\zeta^2}} = 0.02$$

This equation is a conservative estimate, since we are assuming that

$$\cos(\omega_n \sqrt{1-\zeta^2} t - \phi) = 1$$

Settling time

$$T_s = \frac{-\ln(0.02\sqrt{1-\zeta^2})}{\zeta \omega_n}$$

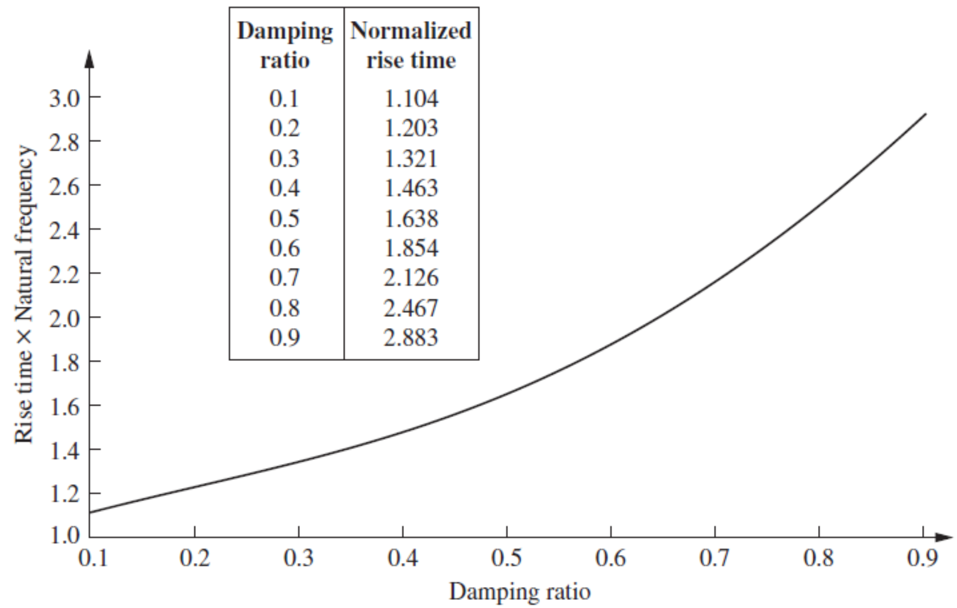
Approximated by

$$T_s = \frac{4}{\zeta \omega_n}$$

Evaluation of T_r

A precise analytical relationship between T_r and ζ cannot be found. However, using a computer, T_r can be found

1. Designate $\omega_n t$ as the normalized time variable
2. Select a value for ζ
3. Solve for the values of $\omega_n t$ that yield $c(t) = 0.9$ and $c(t) = 0.1$
4. The normalized rise time $\omega_n T_r$ is the difference between those two values of $\omega_n t$ for that value of ζ



Damping ratio	Normalized rise time
0.1	1.104
0.2	1.203
0.3	1.321
0.4	1.463
0.5	1.638
0.6	1.854
0.7	2.126
0.8	2.467
0.9	2.883

Figure: Normalized T_r vs. ζ for a 2^{nd}-order underdamped response

Location of poles

- *Natural frequency, ω_n:* Radial distance from the origin to the pole
- *Damping ratio, ζ:* Ratio of the magnitude of the real part of the system poles over the natural frequency

$$\cos(\theta) = \frac{-\zeta\omega_n}{\omega_n} = \zeta$$

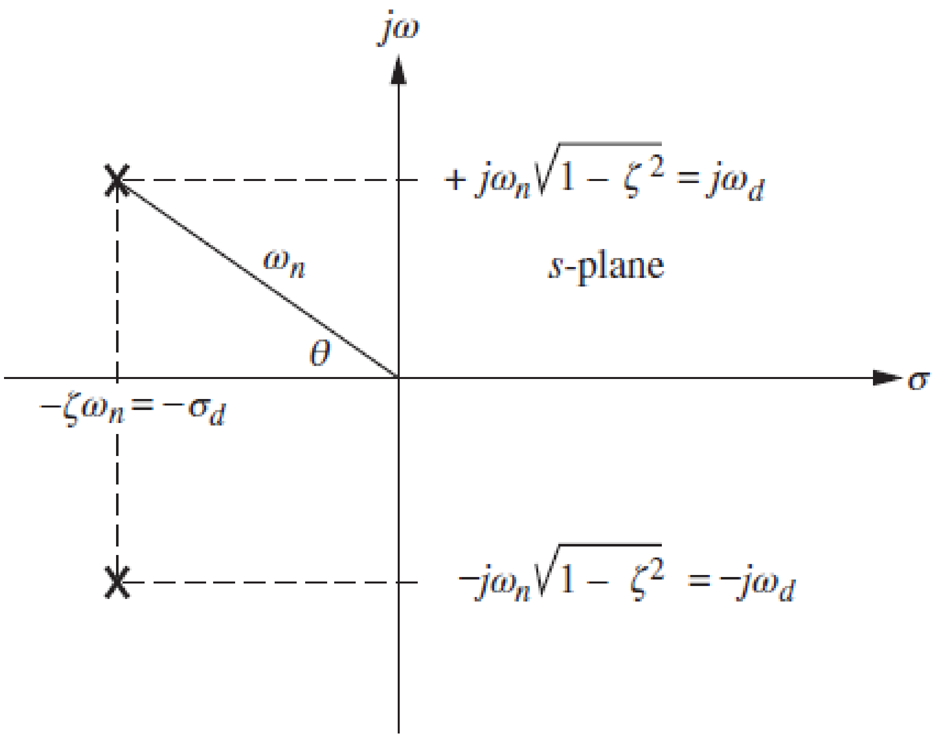


Figure: Pole plot for an underdamped 2^{nd}-order system

Location of poles

- *Damped frequency of oscillation, ω_d:* Imaginary part of the system poles

$$\omega_d = \omega_n \sqrt{1-\zeta^2}$$

- *Exponential damping frequency, σ_d:* Magnitude of the real part of the system poles

$$\sigma_d = \zeta\omega_n$$

- *Poles*

$$s_{1,2} = -\sigma_d \pm j\omega_d$$

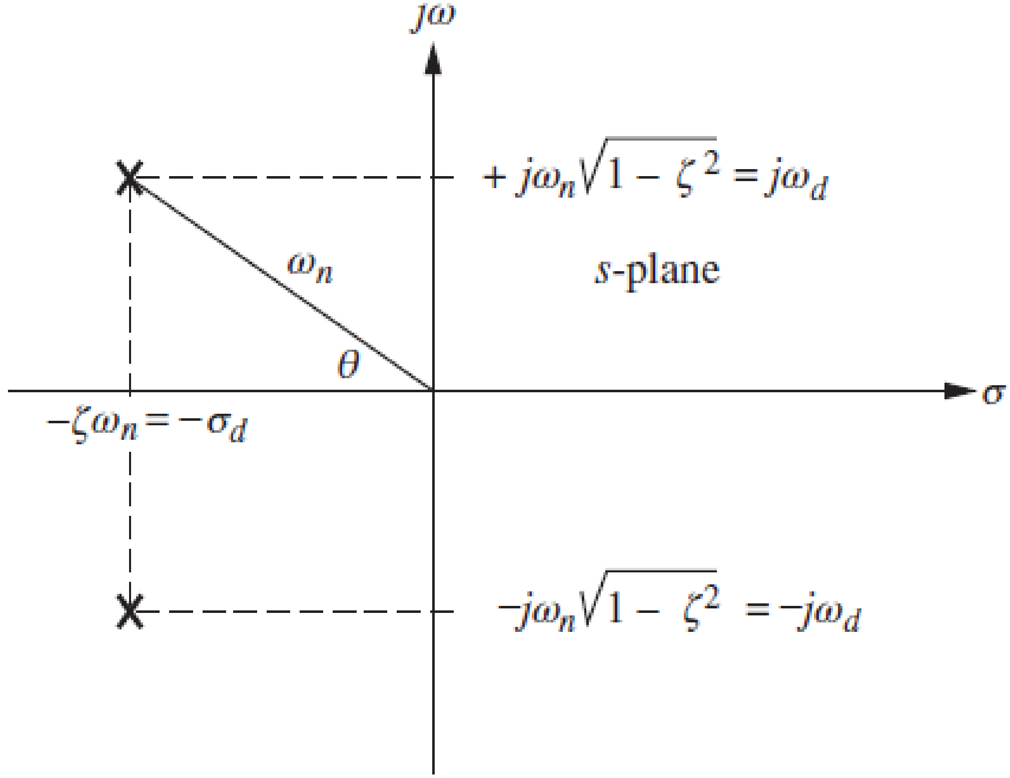


Figure: Pole plot for an underdamped 2^{nd}-order system

Location of poles

- *$T_p \propto$ horizontal lines*

$$T_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} = \frac{\pi}{\omega_d}$$

- *$T_s \propto$ vertical lines*

$$T_s = \frac{4}{\zeta\omega_n} = \frac{4}{\sigma_d}$$

- *$\%OS \propto$ radial lines*

$$\%OS = e^{\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}} \times 100$$

$$\zeta = \cos(\theta)$$

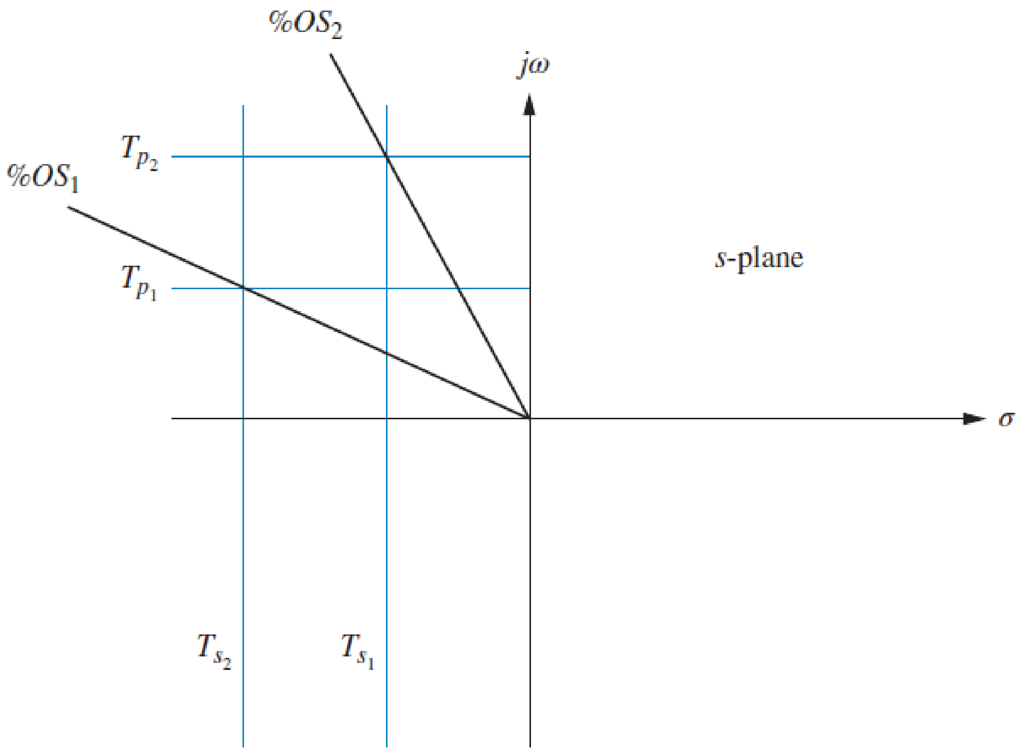


Figure: Lines of constant T_p, T_s, and $\%OS$. Note: $T_{s_2} < T_{s_1}$, $T_{p_2} < T_{p_1}$, $\%OS_1 < \%OS_2$.

Underdamped systems

- $T_p \propto$ *horizontal lines*

$$T_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} = \frac{\pi}{\omega_d}$$

- $T_s \propto$ *vertical lines*

$$T_s = \frac{4}{\zeta\omega_n} = \frac{4}{\sigma_d}$$

- $\%OS \propto$ *radial lines*

$$\%OS = e^{\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}} \times 100$$

$$\zeta = \cos(\theta)$$

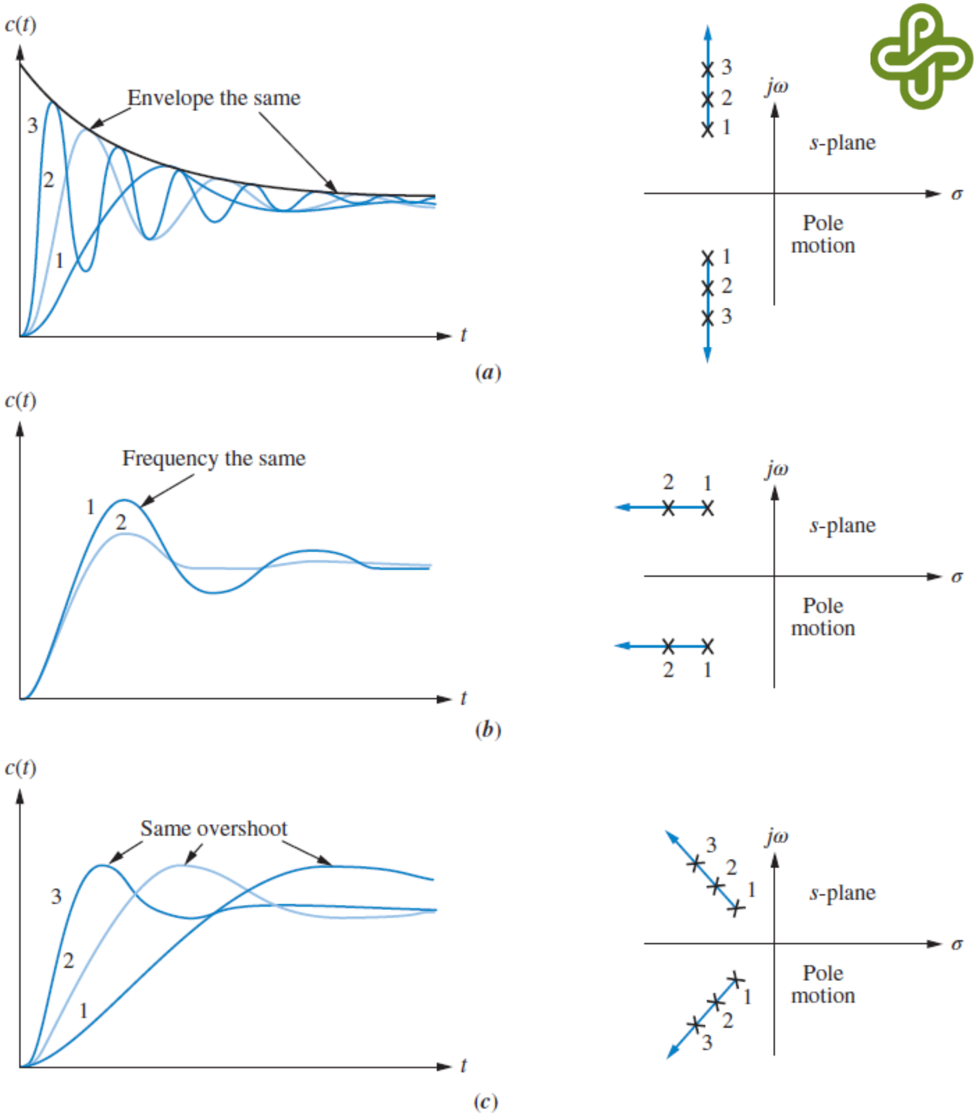


Figure: Step responses of 2^{nd}-order systems as poles move: a. with constant real part, b. with constant imaginary part, c. with constant ζ

Summary

- Transient time response
  - First-order system
    - time constant
    - Specs: rise time, settling time
  - Second-order system
    - Underdamped
    - Critically damped
    - Underdamped
      - Damping ratio, undamped natural frequency: (ζ, ω_n)
      - Specs: rise time, settling time, percentage overshoot, peak time
- Next, modeling the dc-to-dc converter system (the system in the labs)